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k-Ordered Hamilton cycles in digraphs

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Abstract

Given a digraph D, let $\delta^0(D) := \min\{\delta^+(D), \delta^-(D)\}$ be the minimum semi-degree of D. D is k-ordered Hamiltonian if for every sequence $s_1, \ldots, s_k$ of distinct vertices of D there is a directed Hamilton cycle which encounters $s_1, \ldots, s_k$ in this order. Our main result is that every digraph D of sufficiently large order n with $\delta^0(D) \geqslant \lceil (n+k)/2 \rceil - 1$ is k-ordered Hamiltonian. The bound on the minimum semi-degree is best possible. An undirected version of this result was proved earlier by Kierstead, Sárközy and Selkow [H. Kierstead, G. Sárközy, S. Selkow, On k-ordered Hamiltonian graphs, J. Graph Theory 32 (1999) 17–25].

Keywords: Hamilton cycles; Directed graphs; Ordered cycles; Linkedness

1. Introduction

The famous theorem of Dirac determines the smallest minimum degree of a graph which guarantees the existence of a Hamilton cycle. There are many subsequent results which investigate degree conditions that guarantee the existence of a Hamilton cycle with some additional properties. In particular, Chartrand (see [13]) introduced the notion of a Hamilton cycle which has to visit a given set of vertices in a prescribed order. More formally, we say that a graph G is k-*ordered* if for every sequence $s_1, \ldots, s_k$ of distinct vertices of G there is a cycle which encounters $s_1, \ldots, s_k$ in this order. G is k-*ordered Hamiltonian* if it contains a Hamilton cycle with this property. Kierstead, Sárközy and Selkow [10] showed that for all $k \geqslant 2$ every graph on $n \geqslant 11k - 3$ vertices of minimum degree at least $\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$ is k-ordered Hamiltonian. This bound on the minimum degree is best possible and proved a conjecture of Ng and Schultz [13]. Several

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related problems have subsequently been considered: for instance, the case when k is large compared to n was investigated in [6] (but has not been completely settled yet). Ore-type conditions were investigated in [5,6,13]. For more results in this direction, see the survey by Gould [8].

It seems that digraphs provide an equally natural setting for such problems. Our main result is a version of the result in [10] for digraphs. The digraphs we consider do not have loops and we allow at most one edge in each direction between any pair of vertices. Given a digraph D, the *minimum semi-degree* $\delta^0(D)$ of D is the minimum of the minimum outdegree $\delta^+(D)$ of D and its minimum indegree $\delta^-(D)$.

Theorem 1. *For every $k \geqslant 3$ there is an integer $n_0 = n_0(k)$ such that every digraph D on $n \geqslant n_0$ vertices with $\delta^0(D) \geqslant \lceil (n+k)/2 \rceil - 1$ is k-ordered Hamiltonian.*

Our proof shows that one can take $n_0 := Ck^9$ where C is a sufficiently large constant. Note that if n is even and k is odd the bound on the minimum semi-degree is slightly larger than in the undirected case. However, it is best possible in all cases. In fact, if the minimum semi-degree is smaller, it turns out that D need not even be k-ordered. This is easy to see if k is even: let D be the digraph which consists of a complete digraph A of order $\lceil n/2 \rceil + k/2 - 1$ and a complete digraph B of order $\lfloor n/2 \rfloor + k/2$ which has precisely $k-1$ vertices in common with A. Pick vertices $s_1, s_3, \dots, s_{k-1} \in A - B$ and $s_2, s_4, \dots, s_k \in B - A$. Then D has no cycle which encounters $s_1, \dots, s_k$ in this order. A similar construction also works if both k and n are odd. The construction in the remaining case is a little more involved, see [11] for details. Note that every Hamiltonian digraph is 2-ordered Hamiltonian, so the case when $k \leqslant 2$ in Theorem 1 is covered by the result of Ghouila-Houri [7] (Theorem 4 below) which implies that every digraph with minimum semi-degree at least $n/2$ contains a Hamilton cycle.

Theorem 1 can be used to deduce a version for edges which have to be traversed in a prescribed order by the Hamilton cycle: we say that a digraph D is *k-arc ordered Hamiltonian* if, for every sequence $e_1, \dots, e_k$ of independent edges, D contains a Hamilton cycle which encounters $e_1, \dots, e_k$ in this order. D is *k-arc Hamiltonian* if it contains a Hamilton cycle which encounters these edges in any order. D is called *Hamiltonian k-linked* if $|D| \geqslant 2k$ and if for every sequence $x_1, \dots, x_k, y_1, \dots, y_k$ of distinct vertices there are disjoint paths $P_1, \dots, P_k$ in D such that P_i joins x_i to y_i and such that together all the P_i cover all the vertices of D. Thus every digraph D which is Hamiltonian k-linked is also k-arc ordered Hamiltonian. Indeed, if $x_1y_1, \dots, x_ky_k$ are the (directed) edges our Hamilton cycle has to encounter, then disjoint paths linking y_{i-1} to x_i for all $i = 1, \dots, k$ yield the required Hamilton cycle.

Corollary 2. *For all $k \geqslant 3$ there is an integer $n_0 = n_0(k)$ such that every digraph D on $n \geqslant n_0$ vertices with $\delta^0(D) \geqslant \lceil n/2 \rceil + k - 1$ is Hamiltonian k-linked and thus in particular k-arc ordered Hamiltonian.*

The examples in [11] show that in both parts of Corollary 2 the bound on the minimum semi-degree is best possible. In fact, if the minimum semi-degree is smaller, then one cannot even guarantee the digraph to be k-arc ordered. A result of Bermond [3] (see also [2]) implies that if $\delta^0(D) \geqslant \lceil (n+k)/2 \rceil$, then D is k-arc Hamiltonian. It easily follows that if $\delta^0(D) \geqslant \lceil (n+1)/2 \rceil$, then D is Hamiltonian 1-linked, i.e. Hamiltonian connected (see [2]). This covers the case $k = 1$ of Corollary 2. As observed in [1, Theorem 9.2.10], if $\delta^0(D) \geqslant \lceil n/2 \rceil + 1$, then D is Hamiltonian 2-linked, which covers the case $k = 2$ of Corollary 2.

Corollary 2 can easily be deduced from Theorem 1 as follows: let $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ be distinct vertices where we aim to link x_i to y_i for all i. Let D' be the digraph obtained from D by contracting x_i and y_{i-1} into a new vertex s_i whose outneighbourhood is that of x_i and whose inneighbourhood is that of y_{i-1}. More precisely, let $A := \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$. Then D' is the digraph obtained from $D - A$ by adding new vertices $s_1, \ldots, s_k$ and defining the edges incident to these new vertices as follows. The outneighbours of s_i are the outneighbours of x_i in $V(D) \setminus A$ as well as all the s_j for all those $j \neq i-1$ for which y_j is an outneighbour of x_i in D (where $y_0 := y_k$). Similarly, inneighbours of s_i are the inneighbours of y_{i-1} in $V(D) \setminus A$ as well as all the s_j for all those $j \neq i$ for which x_j is an inneighbour of y_{i-1} in D. It is easy to check that $\delta^0(D') \geqslant \lceil (|D'| + k)/2 \rceil - 1$ and that a Hamilton cycle in D' which encounters $s_1, \ldots, s_k$ in this order corresponds to a spanning set of disjoint paths from x_i to y_i.

A result of Chen et al. [4, Theorem 10] implies that the smallest minimum degree which guarantees an undirected graph to be k-arc ordered Hamiltonian is $\lfloor n/2 \rfloor + k - 1$. (A graph is k-arc ordered Hamiltonian if for any sequence of k independent oriented edges there exists a Hamilton cycle which encounters these edges in the given order and orientation.) The smallest minimum degree which forces a graph to be k-linked was determined by Kawarabayashi, Kostochka and Yu [9]. It is not clear whether the minimum degree for Hamiltonian k-linkedness is the same.

The main tool in our proof of Theorem 1 is a recent result by the first authors (Theorem 3 below), which shows that the degree condition in Theorem 1 at least guarantees a k-ordered cycle (but not necessarily a Hamiltonian one). The strategy of the proof of Theorem 1 is to consider such a cycle of maximal length and to show that it must be Hamiltonian. The same strategy was already applied in the proof of the undirected case in [10]. However, both parts of the strategy are more difficult in the digraph case: the existence of a k-ordered directed cycle (i.e. Theorem 3) already confirms a conjecture of Manoussakis [12] for large n. The Hamiltonicity of a k-ordered cycle of maximal length is easier to show in the undirected case as one can consider 'local transformations' of a given k-ordered cycle which reverse the orientation of certain segments of the cycle. This means that apart from some basic observations like Lemma 8 below our proof is quite different from that in [10].

Theorem 3. *(See [11].) Let k and n be integers such that $k \geqslant 2$ and $n \geqslant 200k^3$. Then every digraph D on n vertices with $\delta^0(D) \geqslant \lceil (n+k)/2 \rceil - 1$ is k-ordered.*

2. Notation and tools

Given a digraph D, we write $V(D)$ for its vertex set, $E(D)$ for its edge set and $|D| := |V(D)|$ for its order. We write xy for the edge directed from x to y. More generally, if A and B are disjoint sets of vertices of D, then an *A–B edge* is an edge of the form ab where $a \in A$ and $b \in B$. A digraph is *complete* if every pair of distinct vertices is joined by edges in both directions.

Given disjoint subdigraphs D_1 and D_2 of a digraph D such that $D_1 \cup D_2$ is spanning and a set $A \subseteq V(D_1)$, we write $N^+_{D_i}(A)$ for the set of all those vertices $x \in V(D_i) \setminus A$ which in the digraph D receive an edge from some vertex in A. $N^-_{D_i}(A)$ is defined similarly. If A consists of a single vertex x, we just write $N^+_{D_i}(x)$, etc. and put $d^+_{D_i}(x) := |N^+_{D_i}(x)|$ and $d^-_{D_i}(x) := |N^-_{D_i}(x)|$. So in particular, $N^+_D(x)$ is the outneighbourhood of x in D and $d^+_D(x)$ is its outdegree. Also, note that $N^+_{D_1}(x)$ is the outneighbourhood of x in the subdigraph $D[V(D_1)]$ of D induced by $V(D_1)$ and not its outneighbourhood in D_1 (where $x \in D_1$). We let $N_D(x) := N^+_D(x) \cup N^-_D(x)$.

If we refer to paths and cycles in digraphs, then we always mean that they are directed without mentioning this explicitly. The *length* of a path is the number of its edges. Given two vertices $x, y \in D$, an x–y *path* is a path which is directed from x to y. Given two vertices x and y on a directed cycle C, we write xCy for the subpath of C from x to y. Similarly, given two vertices x and y on a directed path P such that x precedes y, we write xPy for the subpath of P from x to y.

A digraph D is *strongly connected* if for every ordered pair x, y of vertices of D there exists an x–y path. D is *Hamiltonian connected* if for every ordered pair x, y of vertices of D there exists a Hamilton path from x to y. (So Hamiltonian connectedness is the same as Hamiltonian 1-linkedness.)

We will often use the following result of Ghouila-Houri [7] which gives a sufficient condition for the existence of a Hamilton cycle in a digraph. In particular, it implies a version of Theorem 1 for $k \leqslant 2$ as any Hamiltonian digraph is 2-ordered Hamiltonian.

Theorem 4. *Suppose that D is a strongly connected digraph such that $d_D^+(x) + d_D^-(x) \geqslant |D|$ for every vertex $x \in D$. Then D is Hamiltonian.*

The next result of Overbeck-Larisch [14] provides a sufficient condition for a digraph to be Hamiltonian connected.

Theorem 5. *Suppose that D is a digraph such that $d_D^+(x) + d_D^-(y) \geqslant |D| + 1$ whenever xy is not an edge. Then D is Hamiltonian connected.*

3. Preliminary results

Let D be a digraph satisfying the conditions of Theorem 1. Let $S = (s_1, \dots, s_k)$ by any sequence of $k \geqslant 3$ vertices of D. We will often view S as a set. An S-*cycle* in D is a cycle which encounters $s_1, \dots, s_k$ in this order. So we have to show that D has a Hamiltonian S-cycle. Theorem 3 implies the existence of an S-cycle in D. Let C be a longest such cycle and suppose that C is not Hamiltonian. Let H be the subdigraph of D induced by all the vertices outside C. Our aim is to find a longer S-cycle by modifying C (yielding a contradiction). The purpose of this section is to collect the properties of C and H that we need in our proof of Theorem 1.

We let F be the set of all those vertices on C which receive an edge *from* some vertex in H and we let T be the set of all those vertices on C which send an edge *to* some vertex in H. Given $i \in \mathbb{N}$, we write F_i for the set of all those vertices on C which receive an edge from at least i vertices in H. Thus $F_1 = F$. T_i is defined similarly. Given a vertex x on C, we will denote its successor on C by x^+ and its predecessor by x^-.

Lemma 6. *H is Hamiltonian connected and $d_H^-(x) + d_H^+(y) \geqslant |H| + k - 2$ for all vertices $x, y \in H$. Moreover any digraph obtained from H by deleting at most 2 vertices is strongly connected and $k \leqslant |H| \leqslant \lfloor \frac{n-k}{2} \rfloor$.*

Proof. We first show that any two (not necessarily distinct) vertices $x, y \in H$ for which H contains an x–y path, P say, satisfy the degree condition in the lemma. To see this, note that no vertex in $N_C^-(x)$ is a predecessor of some vertex in $N_C^+(y)$. Indeed, if $v \in N_C^-(x)$ and $v^+ \in N_C^+(y)$, then by replacing the edge vv^+ with the path $vxPyv^+$ we obtain a longer S-cycle, a contradiction. But this means that $d_C^-(x) + d_C^+(y) \leqslant |C|$ and thus

$$d_H^-(x) + d_H^+(y) \geqslant 2\left(\left\lceil \frac{n+k}{2} \right\rceil - 1\right) - |C| \geqslant |H| + k - 2, \tag{1}$$

as required. However, as $k \geqslant 3$ this degree condition means that $N_H^-(x) \cap N_H^+(y) \neq \emptyset$ and so H contains a y–x path of length 2. Thus whenever H contains an x–y path it also contains a y–x path.

Now let x and z be any two vertices of H. What we have shown above applied with $y := x$ implies that $d_H^-(x) + d_H^+(x) \geqslant |H| + 1$ and thus $|N_H(x)| \geqslant (|H| + 1)/2$. Note that by the above x is joined to every vertex in $N_H(x)$ with paths in both directions. Similarly, $|N_H(z)| \geqslant (|H| + 1)/2$ and z is joined to every vertex in $N_H(z)$ with paths in both directions. As $|N_H(x) \cap N_H(z)| > 0$ this means that x is joined to z with paths in both directions, i.e. H is strongly connected. Together with (1) this in turn implies that $d_H^-(x) + d_H^+(z) \geqslant |H| + k - 2 \geqslant |H| + 1$ for all vertices $x, z \in H$. In particular, H is Hamiltonian connected by Theorem 5.

To show that any digraph H' obtained from H by deleting at most 2 vertices is strongly connected note that $d_{H'}^-(x) + d_{H'}^+(y) \geqslant |H'| - 1$ for every $x, y \in H'$. Thus if $x \neq y$, then either yx is an edge or H' contains an y–x path of length 2.

It now remains to prove the bounds on $|H|$. Consider any vertex $x \in H$. Then $2(|H| - 1) \geqslant d_H^-(x) + d_H^+(x) \geqslant |H| + k - 2$ and so $|H| \geqslant k$. For the upper bound, note that no vertex in T has a successor in F. Indeed, if v is such a vertex in T and v^+ is its successor, then we could replace vv^+ with a path through H to obtain a longer S-cycle, a contradiction. But this means that some vertex of C must have all its inneighbours on C or all its outneighbours on C. Thus $|C| \geqslant \lceil (n+k)/2 \rceil$ and so $|H| \leqslant \lfloor (n-k)/2 \rfloor$. □

Recall that the proof of Lemma 6 implies the following.

Corollary 7. *No vertex on C which lies in T has a successor in F.*

The next result deals with the case when the vertices $x_1 \in T$ and $x_2 \in F$ are further apart.

Lemma 8. *Suppose that $x_1, x_2 \in C$ are distinct and the interior of x_1Cx_2 does not contain a vertex from S. Then there are no distinct vertices $y_1, y_2 \in H$ such that $x_1y_1, y_2x_2 \in E(D)$.*

Proof. Suppose that such y_1, y_2 do exist. Furthermore, we may assume that x_1 and x_2 are chosen such that they satisfy all these properties and subject to this $|x_1Cx_2|$ is minimum. Let Q denote the set of all vertices in the interior of x_1Cx_2. Then our choice of x_1 and x_2 implies that $N_C^-(y_1) \cap Q = \emptyset$ and $N_C^+(y_2) \cap Q = \emptyset$. Moreover, by Corollary 7 no vertex in $N_C^-(y_1)$ is a predecessor of some vertex in $N_C^+(y_2)$. Thus $d_C^-(y_1) + d_C^+(y_2) \leqslant |C| - |Q| + 1$ and so

$$n + k - 2 \leqslant d_D^-(y_1) + d_D^+(y_2) \leqslant |C| - |Q| + 1 + 2\big(|H| - 1\big) = n - |Q| + |H| - 1.$$

This implies that $|H| > |Q|$ and thus replacing the interior of x_1Cx_2 with a Hamilton path from y_1 to y_2 through H (which exists by Lemma 6) yields a longer S-cycle, a contradiction. □

The next two results will be used in the proof of Lemma 11.

Lemma 9. *Let G be a digraph such that $d^+_G(x) + d^-_G(x) \geqslant |G| + 3$ for every vertex $x \in G$ and $d^+_G(x) + d^-_G(y) \geqslant |G| + 1$ for every pair of vertices $x, y \in G$. Let z_1 and z_2 be distinct vertices of G such that $z_1z_2 \notin E(G)$. Then there exists a vertex $a \in N^+_G(z_1) \cap N^-_G(z_2)$ such that $G - \{z_1, z_2, a\}$ is strongly connected.*

Proof. First note that $|N^+_G(z_1) \cap N^-_G(z_2)| \geqslant 3$ since $z_1z_2 \notin E(G)$. Pick $a_1, a_2, a_3 \in N^+_G(z_1) \cap N^-_G(z_2)$. We will show that one of these a_i can play the role of a. Let $G^* := G - \{z_1, z_2\}$. Note that $d^+_{G^*}(x) + d^-_{G^*}(x) \geqslant |G^*| + 1$ for every vertex $x \in G^*$ and $d^+_{G^*}(x) + d^-_{G^*}(y) \geqslant |G^*| - 1$ for every pair of vertices $x, y \in G^*$. In particular, the latter condition implies that G^* is strongly connected. Thus G^* has a Hamilton cycle C by Theorem 4. Let a^+_1 denote the successor of a_1 on C and let a^-_1 be its predecessor. Put $N^+ := N^+_{G^*}(a^-_1) \setminus \{a_1\}$ and $N^- := N^-_{G^*}(a^+_1) \setminus \{a_1\}$. Note that $|N^+|, |N^-| \geqslant 1$ since $d^+_{G^*}(a^-_1) + d^-_{G^*}(a^-_1) \geqslant |G^*| + 1$ and $d^+_{G^*}(a^+_1) + d^-_{G^*}(a^+_1) \geqslant |G^*| + 1$. Similarly $|N^+| + |N^-| \geqslant |G^*| - 3$. Clearly, if $a^-_1a^+_1$ is an edge or $N^+ \cap N^- \neq \emptyset$, then $G^* - a_1$ is strongly connected and so we can take a to be a_1. So we may assume that neither of these is the case. But then $N^+ \cup N^- = V(G^*) \setminus \{a_1, a^+_1, a^-_1\}$. Let $v \in N^+$ be such that $|vCa^-_1|$ is maximal. Similarly, let $w \in N^-$ be such that $|a^+_1Cw|$ is maximal. Note that if $w \in vCa^-_1$, then $G^* - a_1$ is strongly connected. So we may assume that this is not the case. But then v must be the successor of w on C, N^+ must consist of precisely the vertices in $V(vCa^-_1) \setminus \{a^-_1\}$ and N^- must consist of precisely the vertices in $V(a^+_1Cw) \setminus \{a^+_1\}$.

Let $A^+ := N^+ \cup \{a^-_1\}$ and $A^- := N^- \cup \{a^+_1\}$. We may assume that G does not contain an A^+–A^- edge as otherwise $G^* - a_1$ is strongly connected. We will now show $G^*[A^+]$ is complete and that a_1 receives an edge from every vertex in A^+. So consider any vertex $x \in A^+$. Then $d^+_{G^*}(x) + d^-_{G^*}(a^+_1) \geqslant |G^*| - 1$. Together with the fact that there is no A^+–A^- edge this shows that $N^+_{G^*}(x) = (A^+ \cup \{a_1\}) \setminus \{x\}$. Thus $G^*[A^+]$ is complete and a_1 receives an edge from every vertex in A^+. Similarly one can show that $G^*[A^-]$ is complete and that a_1 sends an edge to every vertex in A^-.

Now consider a_2 and a_3. If for example $a_2 \neq v, w$, then $G^* - a_2$ is strongly connected and so we can take a to be a_2. As one can argue similarly for a_3, we may assume that $v = a_2$ and $w = a_3$. If $a^+_1a^-_1$ is an edge or $a_1 \in N^+_{G^*}(a^+_1) \cap N^-_{G^*}(a^-_1)$, then $G^* - a_2$ is strongly connected. (Here we used that $a^-_1 \neq v = a_2$ since $|N^+| \geqslant 1$.) If this is not the case, then $d^+_{G^*}(a^+_1) + d^-_{G^*}(a^-_1) \geqslant |G^*| - 1$ implies the existence of some vertex $x \in N^+_{G^*}(a^+_1) \cap N^-_{G^*}(a^-_1)$ with $x \neq a_1$. If $x \in A^+$, then a^+_1x is an A^-–A^+ edge avoiding $w = a_3$ and so $G^* - a_3$ is strongly connected. (Here we used that $a^+_1 \neq w = a_3$ since $|N^-| \geqslant 1$.) Similarly, if $x \in A^-$, then $G^* - a_2$ is strongly connected. Altogether, this shows that we can take a to be a_1, a_2 or a_3. □

Lemma 10. *Suppose that H contains a vertex v with $d^-_H(v) + d^+_H(v) \leqslant |H| + k - 1$. Suppose that $x_1, x_2 \in T$ and $y_1, y_2 \in F$ are distinct vertices on C. Then $x_1v, vy_1 \in E(D)$ or $x_2v, vy_2 \in E(D)$ (or both).*

Proof. Let F_v denote the set of all those vertices on C which receive an edge from v. Let T^+_v denote the set of all those vertices on C whose predecessor sends an edge to v. Corollary 7 implies that $T^+_v \cap F_v = \emptyset$. Since

$$d^-_C(v) + d^+_C(v) \geqslant 2\left(\left\lceil \frac{n+k}{2} \right\rceil - 1\right) - \big(|H| + k - 1\big) \geqslant |C| - 1$$

this shows that at most one vertex on C lies outside $T_v^+ \cup F_v$. Let z be the vertex in $V(C) \setminus (T_v^+ \cup F_v)$ (if it exists).

Suppose first that $z \notin F$ (this also covers the case when z does not exist). Then $z \neq y_1, y_2$. Also either $z \neq x_1^+$ or $z \neq x_2^+$. So let us assume that $z \neq x_1^+$ (the case when $z \neq x_2^+$ is similar). We will show that $x_1 v, v y_1 \in E(D)$. So suppose first that $x_1 v \notin E(D)$. Then $x_1^+ \notin T_v^+$ and thus $x_1^+ \in F_v$, a contradiction to Corollary 7. Similarly, if $v y_1 \notin E(D)$, then $y_1 \notin F_v$ and thus $y_1 \in T_v^+$, i.e. the predecessor of y_1 lies in T, contradicting Corollary 7.

So suppose next that $z \in F$ and thus, by Corollary 7, the predecessor of z does not lie in T. This in turn implies that $z \neq x_1^+, x_2^+$. Moreover either $z \neq y_1$ or $z \neq y_2$. So let us assume that $z \neq y_1$. Similarly as before one can show that $x_1 v, v y_1 \in E(D)$. □

In our proof of Theorem 1 we will frequently need two disjoint paths through H joining two given disjoint pairs of vertices on C in order to modify C into a longer S-cycle. The following lemma implies the existence of such paths provided that the pairs consist of vertices having sufficiently many neighbours in H (see also Corollary 12).

Lemma 11. *Suppose that $X_1, X_2 \subseteq T$ and $Y_1, Y_2 \subseteq F$ are disjoint subsets of $V(C)$ such that $|N_H^+(X_1)|, |N_H^+(X_2)| \geqslant 3$ and $|N_H^-(Y_1)|, |N_H^-(Y_2)| \geqslant 3$. Then there are disjoint X_i–Y_i paths P_i of length at least 2 and such that all inner vertices of P_1 and P_2 lie in H. Moreover, if $|H| \geqslant 15$ and if we even have that $|N_H^+(X_1)|, |N_H^+(X_2)| \geqslant 8$ and $|N_H^-(Y_1)|, |N_H^-(Y_2)| \geqslant 8$, then we can find such paths which additionally satisfy $|P_1 \cup P_2| \geqslant |H|/6$.*

Proof. By disregarding some neighbours if necessary we may assume that $|N_H^+(X_1)| = |N_H^+(X_2)| = |N_H^-(Y_1)| = |N_H^-(Y_2)|$. Our first aim is to show that for some $i \in \{1, 2\}$ there is an X_i–Y_i path P_i which satisfies the following properties:

(i) The graph $H' := H - V(P_i)$ has a Hamilton cycle C'.
(ii) All $x, y \in H'$ satisfy $d_{H'}^+(x) + d_{H'}^-(y) \geqslant |H'| - 2$.
(iii) $3 \leqslant |P_i| \leqslant 5$, i.e. P_i contains at least 1 and at most 3 vertices from H.
(iv) If $i = 1$, then $|N_H^+(X_2) \cap V(P_1)| \leqslant 2$ and $|N_H^-(Y_2) \cap V(P_1)| \leqslant 2$. If $i = 2$, then $|N_H^+(X_1) \cap V(P_2)| \leqslant 2$ and $|N_H^-(Y_1) \cap V(P_2)| \leqslant 2$.

If we have found such an i, say $i = 1$, then our aim is to use the Hamilton cycle C' in order to find P_2. To prove the existence of such an i, recall that Lemma 6 implies $d_H^-(x) + d_H^+(y) \geqslant |H| + k - 2 \geqslant |H| + 1$ for every pair of vertices $x, y \in H$. Thus condition (ii) will hold automatically if (iii) holds.

Now suppose first that there exists a vertex $z_1 \in N_H^+(X_1) \cap N_H^-(Y_1)$. Take $i = 1$ and take P_1 to be any X_1–Y_1 path whose interior consists precisely of z_1. Then $d_{H'}^-(x) + d_{H'}^+(x) \geqslant |H'|$ for every $x \in H'$. As H' is strongly connected by Lemma 6 we can apply Theorem 4 to find a Hamilton cycle C' of H'. (If $|H'| = 2$, then C' will consist of just a double edge.) In the case when $N_H^+(X_2) \cap N_H^-(Y_2) \neq \emptyset$ we proceed similarly.

Now suppose that $N_H^+(X_1) \cap N_H^-(Y_1) = \emptyset$ and $N_H^+(X_2) \cap N_H^-(Y_2) = \emptyset$. Then Lemma 10 implies that $d_H^-(x) + d_H^+(x) \geqslant |H| + k \geqslant |H| + 3$ for every $x \in H$. If there is an $N_H^+(X_1)$–$N_H^-(Y_1)$ edge $z_1 z_2$ take $i := 1$ and take P_1 to be any X_1–Y_1 path whose interior consists of this edge. Then $d_{H'}^-(x) + d_{H'}^+(x) \geqslant |H| - 1 = |H'| + 1$ for every $x \in H'$ and so again, as H' is strongly connected by Lemma 6, we can apply Theorem 4 to find a Hamilton cycle C' of H'. In the case when there is an $N_H^+(X_2)$–$N_H^-(Y_2)$ edge we proceed similarly.

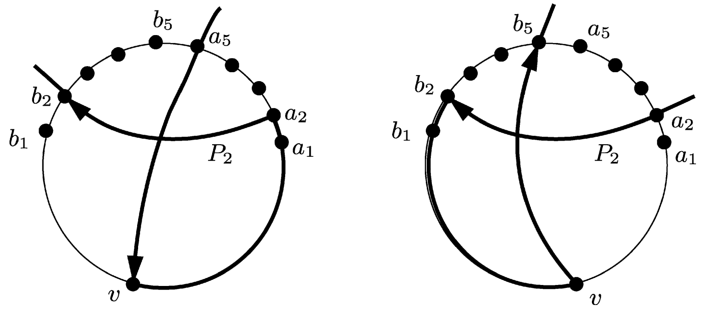

Fig. 1. The path P_2 in Case 1. The left figure is for the subcase when $d^+_{H'}(a_5) \geqslant |H'|/2 - 1$ and the right figure is for the subcase when $d^-_{H'}(b_5) \geqslant |H'|/2 - 1$.

Thus we may assume that $N^+_H(X_i) \cap N^-_H(Y_i) = \emptyset$ and that there is no $N^+_H(X_i)$–$N^-_H(Y_i)$ edge (for $i = 1, 2$). Pick any vertex $z_1 \in N^+_H(X_1)$ and let $z_2 \in N^-_H(Y_1)$ be a vertex such that $|N^+_H(X_2) \cap \{z_1, z_2\}| \leqslant 1$ and $|N^-_H(Y_2) \cap \{z_1, z_2\}| \leqslant 1$. (The fact that we can choose such a z_2 follows from $N^+_H(X_i) \cap N^-_H(Y_i) = \emptyset$ and our assumption that the sizes of the $N^+_H(X_i)$ and the $N^-_H(Y_i)$ are equal.)Apply Lemma 9 with $G := H$ to find a vertex $z_3 \in N^+_H(z_1) \cap N^-_H(z_2)$ such that $H - \{z_1, z_2, z_3\}$ is strongly connected. Take $i := 1$ and P_1 to be any X_1–Y_1 path whose interior consists of $z_1z_3z_2$. Then $d^-_{H'}(x) + d^+_{H'}(x) \geqslant |H| - 3 = |H'|$ for every $x \in H'$ and so again H' contains a Hamilton cycle C' by Theorem 4. Our choice of z_1 and z_2 implies that (iv) holds.

Altogether, this shows that in each case for some i there exists a path P_i satisfying (i)–(iv). We may assume that $i = 1$. As mentioned before, our aim now is to use the Hamilton cycle C' of H' in order to find an X_2–Y_2 path P_2 through H'. In the case when $|N^+_H(X_2)|, |N^-_H(Y_2)| \geqslant 3$ this is trivial since by (iv) both $N^+_H(X_2)$ and $N^-_H(Y_2)$ meet H' in at least one vertex.

So suppose now that $|H| \geqslant 15$ and $|N^+_H(X_2)|, |N^-_H(Y_2)| \geqslant 8$ and thus we wish to find a long X_2–Y_2 path. To do this, let $N^+ := N^+_H(X_2) \cap V(H')$ and $N^- := N^-_H(Y_2) \cap V(H')$. Thus $|N^+|, |N^-| \geqslant 6$ by (iv). Choose $a_1 \in N^+$ and $b_1 \in N^-$ to be distinct such that $|a_1C'b_1|$ is maximum. If $|a_1C'b_1| \geqslant |H'|/6$, then we can take P_2 to be any X_2–Y_2 path whose interior consists of $a_1C'b_1$. So we may assume that $|a_1C'b_1| \leqslant |H'|/6$.

Note that the choice of a_1 and b_1 implies that $N^+, N^- \subseteq V(a_1C'b_1)$. Moreover, all the vertices in N^+ must precede the vertices in N^- on $a_1C'b_1$. (Indeed, if e.g. $a \in N^+$ and $b \in N^-$ are distinct vertices such that b precedes a, i.e. a lies on $bC'b_1$, then $|aC'b| \geqslant |H'| - |a_1C'b_1| \geqslant |H'|/2$, contradicting the choice of a_1 and b_1.) Thus $|N^+ \cap N^-| \leqslant 1$ and there are vertices $a_2, \dots, a_5 \in N^+$ and $b_2, \dots, b_5 \in N^-$ such that $a_1, \dots, a_5, b_5, \dots, b_1$ are distinct and appear on C' in this order. We now distinguish several cases.

Case 1. There are $i, j \leqslant 4$ such that a_ib_j is an edge.

Note that $d^+_{H'}(a_5) \geqslant |H'|/2 - 1$ or $d^-_{H'}(b_5) \geqslant |H'|/2 - 1$ by (ii). Suppose that the former holds (the other case is similar). As $|a_1C'b_1| \leqslant |H'|/6$ this means that a_5 has at least $|H'|/3 - 1$ outneighbours in the interior of $b_1C'a_1$ and so we can find such an outneighbour v with $|vC'a_1| \geqslant |H'|/3$. But then we can take P_2 to be any X_2–Y_2 path whose interior consists of $a_5vC'a_ib_j$ (Fig. 1).

Case 2. For all $i, j \leqslant 4$ a_ib_j is not an edge.

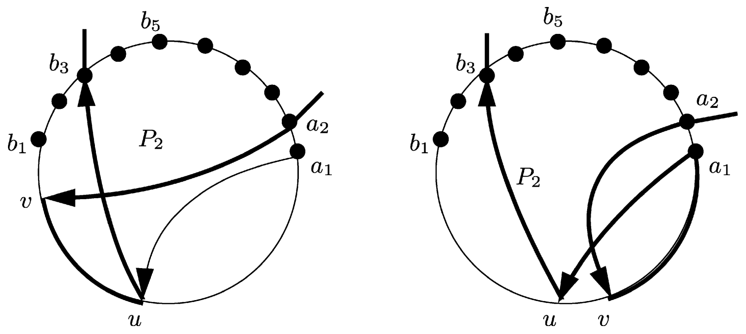

Fig. 2. The path P_2 in Case 2.1 if u lies in the interior of $b_3C'a_1$. The left figure is for the subcase when the interior of $b_3C'u$ contains at least $|H'|/6 - 1$ outneighbours of a_2. The right figure is for the subcase when the interior of $uC'a_1$ contains at least $|H'|/6 - 1$ outneighbours of a_2.

Case 2.1. There exists some vertex $u \in N^+_{H'}(a_1) \cap N^-_{H'}(b_3)$.

Note that $u \neq a_2, b_4$ since by our assumption neither a_2b_3 nor a_1b_4 is an edge. As before, either $d^+_{H'}(a_2) \geqslant |H'|/2 - 1$ or $d^-_{H'}(b_4) \geqslant |H'|/2 - 1$. Suppose that the former holds (the other case is similar).

If u lies in the interior of $a_1C'b_3$, let v be an outneighbour of a_2 in the interior of $b_3C'a_1$ with $|vC'a_1| \geqslant |H'|/3$. Then we can take P_2 to be any X_2–Y_2 path whose interior consists of $a_2vC'a_1ub_3$.

So we may assume that u lies in the interior of $b_3C'a_1$. But then either the interior of $b_3C'u$ contains at least $|H'|/6 - 1$ outneighbours of a_2 or the interior of $uC'a_1$ contains at least $|H'|/6 - 1$ outneighbours of a_2. If the former holds let v be any outneighbour of a_2 in the interior of $b_3C'u$ such that $|vC'u| \geqslant |H'|/6$ and take P_2 to be any X_2–Y_2 path whose interior consists of $a_2vC'ub_3$ (see Fig. 2). If the latter holds let v be any outneighbour of a_2 in the interior of $uC'a_1$ such that $|vC'a_1| \geqslant |H'|/6$ and take P_2 to be any X_2–Y_2 path whose interior consists of $a_2vC'a_1ub_3$.

Case 2.2. There exists some vertex $u \in N^+_{H'}(a_3) \cap N^-_{H'}(b_1)$.

This case is similar to Case 2.1 and we omit the details.

Case 2.3. Both $N^+_{H'}(a_1) \cap N^-_{H'}(b_3)$ and $N^+_{H'}(a_3) \cap N^-_{H'}(b_1)$ are empty.

Together with (ii) and our assumption that a_1b_3 is not an edge this implies that $N^+_{H'}(a_1) \cup N^-_{H'}(b_3) = V(H') \setminus \{a_1, b_3\}$. Since a_3b_3 is not an edge this means that a_1a_3 is an edge. Similarly it follows that b_3b_1 is an edge. But as before either $d^+_{H'}(a_2) \geqslant |H'|/2 - 1$ or $d^-_{H'}(b_2) \geqslant |H'|/2 - 1$. Suppose that the former holds (the other case is similar). Then we can find an outneighbour v of a_2 in the interior of $b_1C'a_1$ with $|vC'a_1| \geqslant |H'|/3$. But then we can take P_2 to be any X_2–Y_2 path whose interior consists of $a_2vC'a_1a_3C'b_1$. $\square$

Lemma 11 immediately implies the following corollary, which is sometimes more convenient to apply.

Corollary 12. *Suppose that $x_1, x_2 \subseteq T_3$ and $y_1, y_2 \subseteq F_3$ are distinct vertices on C. Then D contains disjoint x_i–y_i paths P_i of length at least 2 such that all inner vertices of P_1 and P_2 lie in H. Moreover, if $|H| \geqslant 15$ and if we even have that $x_1, x_2 \subseteq T_8$ and $y_1, y_2 \subseteq F_8$, then we can find such paths which additionally satisfy $|P_1 \cup P_2| \geqslant |H|/6$.*

The last of our preliminary results gives a lower bound on the sizes of T_3 and F_3.

Lemma 13. *We have that $|T|, |F| \geqslant (n+k)/2 - |H|$. Moreover, $|T_3|, |F_3| \geqslant (n-k)/2 - |H|$ and $|T_3 \cup F_3| \geqslant |C| - |H| - 2k$.*

Proof. To see the bound on $|T|$, note that $d_C^-(x) \geqslant \delta^0(D) - (|H| - 1) \geqslant (n+k)/2 - |H|$ for every vertex $x \in H$ and so $|T| \geqslant (n+k)/2 - |H|$. The proof for $|F|$ is similar. To prove the bound on $|T_3|$, we double-count the number $e(T, H)$ of edges in D from T to $V(H)$. Since $d_C^-(x) \geqslant (n+k)/2 - |H|$ for any vertex $x \in H$ we have that $e(T, H) \geqslant |H|((n+k)/2 - |H|)$. On the other hand $e(T, H) \leqslant |T_3||H| + 2(|T| - |T_3|) = |T_3|(|H| - 2) + 2|T|$. Before we can use this to estimate $|T^3|$, we need an upper bound on $|T|$. For this, recall that $|F| \geqslant (n+k)/2 - |H|$. Together with Corollary 7 this shows that $|T| \leqslant |C| - |F| \leqslant (n-k)/2$. Altogether this gives

$$\begin{aligned} |T_3| &\geqslant \frac{|H|((n+k)/2 - |H|) - (n-k)}{|H| - 2} = \frac{(|H| - 2)(n-k)/2 - |H|(|H| - k)}{|H| - 2} \\ &\geqslant \frac{n-k}{2} - |H| = \frac{|C| - |H| - k}{2}. \end{aligned}$$

The proof for $|F_3|$ is similar. The bound on $|T_3 \cup F_3|$ follows since $|T_3 \cap F_3| \leqslant k$. Indeed, the latter holds since Lemma 8 implies that whenever $s, s' \in S$ are distinct and no vertex from S lies in the interior of sCs', then $T_3 \cap F_3$ meets sCs' in at most one vertex. □

4. Proof of Theorem 1

Throughout this section, we assume that the order n of our given digraph D is sufficiently large compared to k for our estimates to hold. We will also omit floors and ceilings whenever this does not affect the argument. Let S, C and H be as defined at the beginning of Section 3. Recall that we assume that C is not Hamiltonian and will show that we can extend C into a longer S-ordered cycle (which would yield a contradiction and thus would prove Theorem 1). Given consecutive vertices $s, s' \in S$, we call the path obtained from sCs' by deleting s' the *interval from s to s'*. Thus no vertex from S lies in the interior of sCs' and C consists of precisely $|S| = k$ disjoint intervals. In our proof of Theorem 1 we distinguish the following 4 cases according to the order of H. Recall that $|H| \geqslant k$ by Lemma 6.

Case 1. $k \leqslant |H| \leqslant 220k^3$.

Recall that $|T_3| \geqslant (n-k)/2 - |H| \geqslant n/3$ by Lemma 13 and so at least one of the k intervals of C must contain at least $n/(3k)$ vertices from T_3. Suppose that this is the case for the interval I from s to s'. Recall that by Lemma 13 at most $|H| + 2k \leqslant 3|H|$ vertices of C do not lie in $T_3 \cup F_3$ and by Corollary 7 no vertex in F_3 is the successor of a vertex in T_3. Since every maximal subpath of I consisting of vertices from T_3 is succeeded by at least one vertex outside $T_3 \cup F_3$, it follows that I contains a subpath A which consists entirely of vertices from T_3 and satisfies

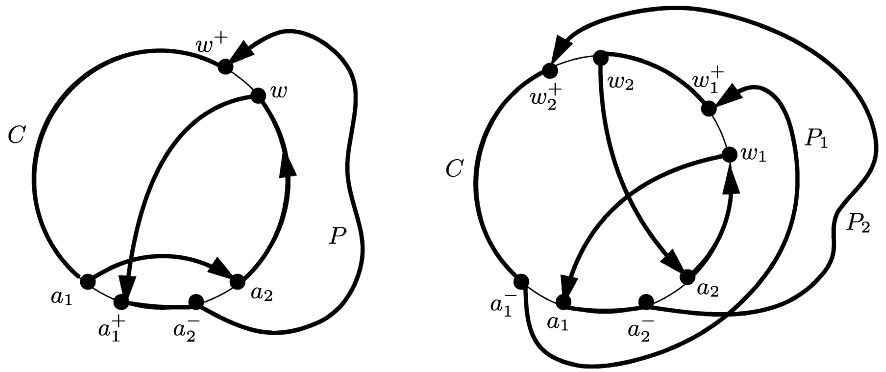

Fig. 3. Extending C into a longer S-ordered cycle in Case 1.1 (left) and Case 1.2 (right).

$|A| \geqslant n/(3k(3|H|+1))$. Let A_1 be the subpath of A consisting of its initial $n/(20k|H|)$ inner vertices and let A_2 be the subpath of A consisting of its last $n/(20k|H|)$ inner vertices.

Let t be the first vertex of A. (So t^+ is the first vertex of A_1.) Consider any vertex a on t^+Cs'. Lemma 8 implies that $a \notin F$. Thus $N_D^-(a) \subseteq V(C)$ and hence

$$d_C^-(a) \geqslant \delta^0(D) \geqslant (n+k)/2 - 1 \geqslant n + k - 1 - |H| - |F| > |C| - |F|. \tag{2}$$

(To see the third inequality recall that $|F| \geqslant (n+k)/2 - |H|$ by Lemma 13.)

Case 1.1. There are vertices $a_1 \in A_1$ and $a_2 \in A_2$ such that a_1a_2 is an edge.

Inequality (2) applied with $a := a_1^+$ implies that there exists a vertex $w \in N_C^-(a_1^+)$ such that the successor w^+ of w lies in F. Recall that F avoids t^+Cs' and so w^+ must lie in $s'Ct - s'$. Hence w must lie in $s'Ct - t$ (and thus in the interior of a_2Ca_1). As $a_2^- \in V(A) \subseteq T_3$ and as H is Hamiltonian connected by Lemma 6, there is an a_2^-–w^+ path P whose interior consists of precisely all the vertices in H. But then the S-ordered cycle $a_1a_2Cwa_1^+Ca_2^-Pw^+Ca_1$ is Hamiltonian, contradicting the choice of C (see Fig. 3).

Case 1.2. There are no such vertices $a_1 \in A_1$ and $a_2 \in A_2$.

Let F_3^- denote the set of all predecessors of vertices in F_3. Recall that F avoids t^+Cs'. Thus F_3^- avoids $tCs' - s'$. Now consider any vertex a on A_2. Then $N_D^-(a) \subseteq V(C)$ since $a \notin F$ and thus $N_D^-(a) \subseteq V(C) \setminus V(A_1)$ by our assumption. But then using that $|F_3| \geqslant (n-k)/2 - |H|$ by Lemma 13 and arguing similarly as in (2) one can show that $d_{C-A_1}^-(a) \geqslant n - 1 - |H| - |F_3| = |C| - 1 - |F_3| = |C - A_1| - |F_3^-| + |A_1| - 1$. Together with the fact that $F_3^- \cap V(A_1) = \emptyset$ this gives

$$\left|N_{C-A_1}^-(a) \cap F_3^-\right| \geqslant |A_1| - 1 \geqslant n/\big(21k|H|\big). \tag{3}$$

Let I_1 be the subpath of the interval I preceding the first vertex in A_1. So $I_1 = sCt$. Let $I_2, \dots, I_k$ denote all the other intervals. For each $i = 1, \dots, k$ let G_i be the auxiliary bipartite graph whose vertex classes are $V(A_2)$ and $V(I_i) \cap F_3^-$ and in which $a \in V(A_2)$ is joined to $w \in V(I_i) \cap F_3^-$ if $wa \in E(D)$. Recall that F_3^- avoids $tCs' - s'$. Thus $F_3^- \subseteq V(I_1) \cup \dots \cup V(I_k)$ and so the edges of $G_1 \cup \cdots \cup G_k$ correspond to the edges of D from F_3^- to A_2. Together with (3) this implies that there is some i such that

$$e(G_i) \geqslant \frac{n|A_2|}{21k^2|H|} \geqslant \frac{n^2}{420k^3|H|^2} \geqslant 3n \geqslant 3|G_i|.$$

Thus G_i is not planar and so there are vertices $a_1, a_2 \in V(A_2)$ and $w_1, w_2 \in V(I_i) \cap F_3^-$ such that the edges w_1a_1, w_2a_2 'cross' in G_i, i.e. such that w_1 lies in the interior of a_2Cw_2 and a_1 lies in the interior of w_2Ca_2. Recall that $w_1^+, w_2^+ \in F_3$ by the definition of F_3^- and $a_1^-, a_2^- \in T_3$ as A_2 consisted of inner vertices of A. Thus we can apply Corollary 12 to obtain disjoint a_j^-–w_j^+ paths P_j having all their inner vertices in H and such that each P_j contains at least one inner vertex (where $j = 1, 2$). Thus $a_1^- P_1 w_1^+ C w_2 a_2 C w_1 a_1 C a_2^- P_2 w_2^+ C a_1^-$ is an S-ordered cycle with at least $|C| + 2$ vertices (note that it contains all the vertices of C), contradicting the choice of C (see Fig. 3).

Case 2. $220k^3 \leqslant |H| \leqslant n/2 - n/(50k)$.

The argument for this case is similar to that in Case 1. Recall that $|T_3| \geqslant (n-k)/2 - |H| \geqslant n/(60k)$ by Lemma 13 and so one of the k intervals of C must contain at least $n/(60k^2)$ vertices from T_3. Suppose that this is the case for the interval I from s to s'. Let t be the first vertex on I that lies in T_3. Let A be the set consisting of the last $n/(70k^2)$ vertices from T_3 lying in the interior of I. For each $a \in A$ let Q_a be the set of $220k^3$ vertices of C preceding a. Note that the definition of A implies that Q_a lies in the interior of I and that t precedes the first vertex of Q_a. Together with Lemma 8 this shows that F avoids t^+Cs' and thus all of $Q_a \cup \{a, a^+\}$. In particular, $a \notin F$. Thus $N_D^-(a) \subseteq V(C)$ and so a satisfies (2).

Case 2.1. There is a vertex $a \in A$ for which a^+ receives an edge from some vertex $q \in Q_a$.

Inequality (2) implies that there exists a vertex $w \in N_C^-(a)$ such that the successor w^+ of w lies in F. Note that w lies in the interior of aCq since F avoids $Q_a \cup \{a, a^+\}$. As $a \in A \subseteq T_3$ and as H is Hamiltonian connected by Lemma 6, there is an a–w^+ path P whose interior consists precisely of all the vertices in H. But then the cycle qa^+CwaPw^+Cq is S-ordered and contains $|H| - |Q_a| + 1 > 0$ more vertices than C, a contradiction.

Case 2.2. There is no such vertex $a \in A$.

This case is similar to Case 1.2. Let F_3^- denote the set of all predecessors of vertices in F_3 again. Let A^+ denote the set of all successors of vertices in A. Recall that F avoids t^+Cs'. Thus F_3^- avoids $tCs' - s'$ and thus in particular all the sets Q_a.

Consider any $a \in A$. Then $N_D^-(a^+) \subseteq V(C)$ since $a^+ \notin F$ by Corollary 7. Thus $N_D^-(a^+) \subseteq V(C) \setminus Q_a$ by our assumption. Hence similarly as in Case 1.2 one can show that $d_{C-Q_a}^-(a^+) \geqslant |C - Q_a| - |F_3^-| + |Q_a| - 1$. Together with the fact that $F_3^- \cap Q_a = \emptyset$ this gives

$$\left|N_{C-Q_a}^-\left(a^+\right) \cap F_3^-\right| \geqslant |Q_a| - 1 \geqslant 210k^3. \tag{4}$$

Let I_1 be the subpath of the interval I preceding the first vertex in A^+. Let $I_2, \ldots, I_k$ denote all the other intervals. For each $i = 1, \ldots, k$ let G_i be the auxiliary bipartite graph whose vertex classes are A^+ and $V(I_i) \cap F_3^-$ and in which $a^+ \in A^+$ is joined to $w \in V(I_i) \cap F_3^-$ if wa^+ is an edge of D. Note that $F_3^- \subseteq V(I_1) \cup \cdots \cup V(I_k)$ since F_3^- avoids $tCs' - s'$. Thus the edges of $G_1 \cup \cdots \cup G_k$ correspond to the edges from F_3^- to A^+. Together with (4) this implies that there is some i such that

$$e(G_i) \geqslant \frac{210k^3|A^+|}{k} = \frac{210k^3 n}{70k^3} = 3n \geqslant 3|G_i|.$$

Thus G_i is not planar and so there are vertices $a_1^+, a_2^+ \in V(A^+)$ and $w_1, w_2 \in V(I_i) \cap F_3^-$ such that the edges $w_1a_1^+$, $w_2a_2^+$ cross. As in Case 1.2 we can apply Corollary 12 to obtain disjoint a_j–w_j^+ paths having all their inner vertices in H such that each P_j contains at least one inner vertex (where $j = 1, 2$ and a_j is the predecessor of a_j^+). Thus $a_1 P_1 w_1^+ C w_2 a_2^+ C w_1 a_1^+ C a_2 P_2 w_2^+ C a_1$ is an S-ordered cycle with at least $|C| + 2$ vertices (note that it contains all the vertices of C), contradicting the choice of C.

Case 3. $n/2 - n/(50k) \leqslant |H| \leqslant \lceil (n-k)/2 \rceil - 1$.

Our first aim is to find vertices x_1, x_2, y_1, y_2 on C with the following properties:

(i) x_1, x_2, y_1, y_2 occur on C in this order and either all of these vertices are distinct or else $|\{x_1, x_2, y_1, y_2\}| = 3$ and $x_1 = y_2$.
(ii) S avoids the interior of x_1Cx_2, the interior of y_1Cy_2 as well as x_2 and y_1.
(iii) There are distinct vertices $h_1, h_2, h_1', h_2' \in H$ such that $x_1h_1, x_2h_2, h_1'y_1, h_2'y_2$ are edges.
(iv) If $x_1 \neq y_2$ (and so x_1, x_2, y_1, y_2 are distinct), then there are disjoint x_i–y_i paths P_i of length at least 2 such that all inner vertices of P_1 and P_2 lie in H and $|P_1 \cup P_2| \geqslant |H|/6$.

To prove the existence of such vertices, suppose first that $|T_8| \geqslant k+1$ and $|F_8| \geqslant k+1$. Then we can find two vertices $x_1, x_2 \in T_8$ and two vertices $y_1, y_2 \in F_8$ satisfying (ii). Then these vertices automatically satisfy (iii). Lemma 8 implies that they also satisfy (i). Finally, if they are all distinct, then Corollary 12 shows that they also satisfy (iv).

So suppose next that for example $|T_8| \leqslant k$ but $|F_8| \geqslant k+1$. Pick $y_1, y_2 \in F_8$ as before. To find x_1 and x_2, first note that each vertex $h \in H$ satisfies

$$d_C^-(h) \geqslant \delta^-(D) - \bigl(|H| - 1\bigr) \geqslant \bigl\lceil (n+k)/2 \bigr\rceil - 1 - \bigl\lceil (n-k)/2 \bigr\rceil + 2 = k+1$$

and so h receives at least one edge from some vertex in $T \setminus T_8$. As each vertex in $T \setminus T_8$ sends an edge to at most 7 vertices in H, this means that there are at least $|H|/7$ independent edges from C to H. Thus the interior of some interval of C contains the endvertices of 16 of these independent edges which avoid y_1 and y_2. Let X_1 be the set of the first 8 endvertices of these edges on this interval and let X_2 be the set of the next 8 endvertices. Then Lemma 11 implies that there are vertices $x_1 \in X_1$ and $x_2 \in X_2$ which together with y_1 and y_2 satisfy (iv). By construction, x_1, x_2, y_1, y_2 are all distinct and satisfy (ii) and (iii). Again, Lemma 8 implies that they also satisfy (i). The cases when $|T_8| \geqslant k+1$ but $|F_8| \leqslant k$ and when $|T_8|, |F_8| \leqslant k$ are similar. So we have shown that there are vertices x_1, x_2, y_1, y_2 satisfying (i)–(iv).

In what follows, we will frequently use the fact that any vertex $x \in V(C) \setminus F_2$ receives an edge from all but at most

$$|C| - \bigl(\delta^-(D) - 1\bigr) \leqslant n/2 + n/(50k) - (n+k)/2 + 2 \leqslant n/(45k)$$

vertices of C. Similarly, any vertex $x \in V(C) \setminus T_2$ sends an edge to all but at most $n/(45k)$ vertices of C.

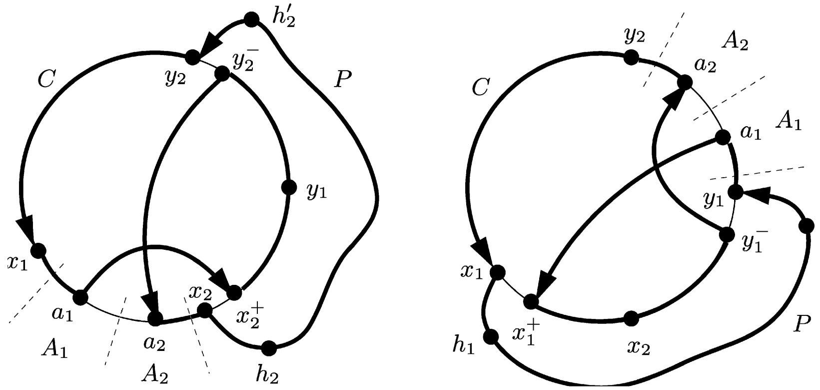

Fig. 4. Extending C into a longer S-ordered cycle in Case 3.1 (left) and Case 3.2 (right).

Case 3.1. $|x_1Cx_2| \geqslant n/(15k)$.

Let A_2 be the set of $n/(40k)$ vertices which immediately precede x_2 and let A_1 be the set of $n/(40k)$ vertices which immediately precede A_2. Corollary 7 implies that the successor x_2^+ of x_2 on C does not lie in F. Thus x_2^+ receives an edge from some vertex $a_1 \in A_1$ since it receives an edge from all but at most $n/(45k)$ vertices of C. Similarly, the predecessor y_2^- of y_2 does not lie in T and thus sends an edge to some vertex $a_2 \in A_2$. Lemma 6 now implies that H contains a Hamilton path P from h_2 to h_2'. But then the cycle $a_1x_2^+Cy_2^-a_2Cx_2h_2Ph_2'y_2Ca_1$ is S-ordered and contains all vertices of C except those in the interior of a_1Ca_2 (see Fig. 4). But as $|H| > n/4 > |a_1Ca_2|$ this means that this new cycle is longer than C, a contradiction.

Case 3.2. $|y_1Cy_2| \geqslant n/(15k)$.

The proof of this case is similar to that of Case 3.1. Let A_1 be the set of $n/(40k)$ vertices which immediately succeed y_1 and let A_2 be the set of $n/(40k)$ vertices which immediately succeed A_1. Then the predecessor y_1^- of y_1 sends an edge to some vertex $a_2 \in A_2$ and the successor x_1^+ of x_1 receives an edge from some vertex $a_1 \in A_1$. Then the S-ordered cycle $y_1^-a_2Cx_1h_1Ph_1'y_1Ca_1x_1^+Cy_1^-$ is longer than C, where P is a Hamilton path in H from h_1 to h_1' (see Fig. 4).

Case 3.3. $|y_2Cx_1| \geqslant n/5$.

Let Z be a segment of the interior of y_2Cx_1 such that $|Z| \geqslant n/(6k)$ and such that Z avoids S. Let Z_1 be the set consisting of the first $n/(40k)$ vertices on Z. Let Z_2 be the set consisting of the next $n/(40k)$ vertices and define $Z_3, \dots, Z_6$ similarly. As by Corollary 7 the predecessor y_1^- of y_1 does not lie in T it must send an edge to some vertex $z_4 \in Z_4$. Similarly the predecessor y_2^- of y_2 sends an edge to some vertex $z_2 \in Z_2$, the successor x_1^+ of x_1 receives an edge from some vertex $z_5 \in Z_5$ and the successor x_2^+ of x_2 receives an edge from some vertex $z_3 \in Z_3$. Now Lemma 8 implies that either $Z_1 \cap T_2 = \emptyset$ or $Z_6 \cap F_2 = \emptyset$ or both. If $Z_1 \cap T_2 = \emptyset$, then every vertex in Z_1 sends an edge to Z_6 (since every vertex outside T_2 sends an edge to all but at most $n/(45k)$ vertices on C). Similarly, if $Z_6 \cap F_2 = \emptyset$, then every vertex in Z_6 receives an edge from some vertex in Z_1. So in both cases we can find a Z_1–Z_6 edge z_1z_6. But then the

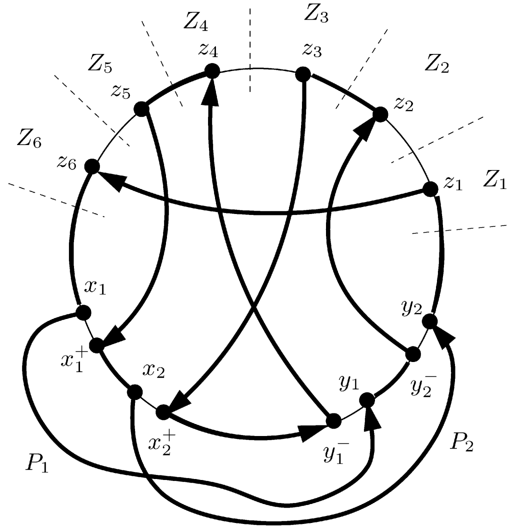

Fig. 5. Extending C into a longer S-ordered cycle in Case 3.3.

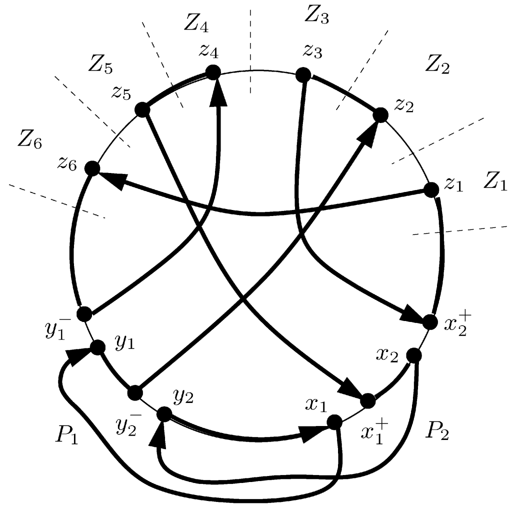

Fig. 6. Extending C into a longer S-ordered cycle in Case 3.4.

cycle $x_1P_1y_1Cy_2^-z_2Cz_3x_2^+Cy_1^-z_4Cz_5x_1^+Cx_2P_2y_2Cz_1z_6Cx_1$ is S-ordered and contains at least $|P_1 \cup P_2| - 4 - (|Z| - 6) \geqslant |H|/6 - n/(6k) > 0$ vertices more than C, a contradiction (see Fig. 5).

Case 3.4. None of Cases 3.1–3.3 holds.

In this case we must have that $|x_2Cy_1| \geqslant n/5$ and can argue similarly as in Case 3.3 (see Fig. 6). We omit the details.

Case 4. None of Cases 1–3 holds.

Together with Lemma 6 this implies that $n-k$ is even and $|H|=(n-k)/2$. So $|C|=(n+k)/2$. First note that any vertex $h \in H$ satisfies

$$d_C^+(h), d_C^-(h) \geqslant (n+k)/2 - 1 - \big(|H|-1\big) = k. \tag{5}$$

Moreover, if $h, h' \in H$ are distinct and if $s \in S \cap N_C^-(h)$, then by Lemma 8 the special vertex s' succeeding s on C (i.e. the unique vertex $s' \in S$ for which S avoids the interior of sCs') cannot lie in $N_C^+(h')$. Thus $|S \cap N_C^-(h)| + |S \cap N_C^+(h')| \leqslant k$ and so

$$\big|N_C^-(h) \setminus S\big| + \big|N_C^+(h') \setminus S\big| \geqslant \big|N_C^-(h)\big| + \big|N_C^+(h')\big| - k \overset{(5)}{\geqslant} k. \tag{6}$$

Case 4.1. There exists some vertex $x \in N_C^-(h) \setminus S$.

First note that by Corollary 7 the successor x^+ of x on C does not lie in F. Thus $d_C^-(x^+) \geqslant \delta^0(D) = |C|-1$ and so x^+ receives an edge from the predecessor x^- of x. Pick any vertex $y \in F \setminus \{x, x^-\}$. (Such a vertex exists since $|F| \geqslant 3$ by (5).) Note that $y \neq x^+$ since $x^+ \notin F$. By Corollary 7 the predecessor y^- of y does not send an edge to H and so y^-x must be an edge (since $d_C^+(y^-) = |C|-1$). Now apply Lemma 6 to find an x–y path P of length at least 2 all whose inner vertices lie in H. Then $x^-x^+Cy^-xPyCx^-$ is an S-ordered cycle which is longer than C, a contradiction.

Case 4.2. There is no vertex as in Case 4.1.

Together with (6) this implies that we can find a vertex $x \in N_C^+(h') \setminus S$. We then argue similarly as in Case 4.1. This completes the proof of Theorem 1.

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